

Affine Multiplexing Networks: System Analysis, Learning, and Computation

Ivan Papusha Ufuk Topcu Steven Carr Niklas Lauffer

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Abstract

We introduce a novel architecture and computational framework for formal, automated analysis of systems with a broad set of nonlinearities in the feedback loop, such as neural networks, vision controllers, switched systems, and even simple programs. We call this computational structure an *affine multiplexing network* (AMN). The architecture is based on interconnections of two basic conceptual building blocks: multiplexers (μ), and affine transformations (α). When attached together appropriately, these building blocks translate to conjunctions and disjunctions of affine statements, resulting in an encoding of the network into satisfiability modulo theory (SMT), mixed integer programming, and sequential convex optimization solvers.

We show how to formulate and verify system properties like stability and robustness, how to compute margins, and how to verify performance through a sequence of SMT queries. As illustration, we use the framework to verify closed loop, possibly nonlinear dynamical systems that contain neural networks in the loop, and hint at a number of extensions that can make AMNs a potent playground for interfacing between machine learning, control, convex and nonconvex optimization, and formal methods.

1 Introduction

1.1 Affine multiplexing networks

This work proposes a novel computational structure called an *affine multiplexing network* (AMN), or an *affine if-then-else network*, which is formed by the composition of multiplexing functions and affine transformations in a dimension compatible way, and parameterized by the weights and biases of the affine transformations.

By repeatedly instantiating and connecting these two components in an acyclic computation graph, we can construct arbitrary relations between the

entire network’s inputs and outputs. The result of such an interconnection is an artificial neural network with interspersed multiplexing nonlinearities. Models built with the AMN viewpoint are much more powerful in practice for certain applications than general neural network models, because as we will see, many mathematical properties of AMNs can be formally and automatically verified with existing tools. Meanwhile, AMNs can be trained just as easily as neural networks.

The AMN model encompasses many other neural networks as a special case, including deep multilayer feedforward networks with piecewise linear nonlinearities, e.g., rectified linear unit (ReLU), absolute value, saturation, dead-zone, and max-pooling nonlinearities. In particular, AMNs can be readily applied to quantifying resilience in classifiers [CNR17], in software safety verification [HKWW17, Ehl17], and in certification [KBD⁺17].

Building blocks Define the *multiplexing* function $\mu : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$,

$$\mu(x, y, z) \triangleq \begin{cases} x, & \text{if } z \leq 0, \\ y, & \text{otherwise.} \end{cases} \quad (1)$$

The function μ represents a ternary *choice* assignment, similar to the operation

$$w := \mu(x, y, z) \iff w := \text{if } z \leq 0 \text{ then } x \text{ else } y.$$

This multiplexing function is the first building block of an AMN, and can be visualized as a 2-to-1 multiplexing unit, shown in the left pane of Figure 1. The value of $\mu(x, y, z)$ is either x or y , depending on whether the statement $z \leq 0$ is true (1) or false (0). Motivated by electronic component nomenclature, we refer to x and y as the *signals* or *inputs*, and z as the *select* or *enable* input. If the value of z satisfies the one-dimensional linear *enable condition* ($z \leq 0$), then the *output* w obtains the value x ; otherwise ($z > 0$) the output w obtains the value y .

The second building block is an *affine transformation*, which is any function, $\alpha : \mathbf{R}^n \rightarrow \mathbf{R}^m$, with a vector input and output, of the form

$$\alpha(x) \triangleq Ax + b, \quad A \in \mathbf{R}^{m \times n}, \quad b \in \mathbf{R}^m. \quad (2)$$

An affine transformation is visualized as an amplifier, shown in the right pane of Figure 1, and parameterized by a *gain* or *weight* matrix A and a *bias* vector b .

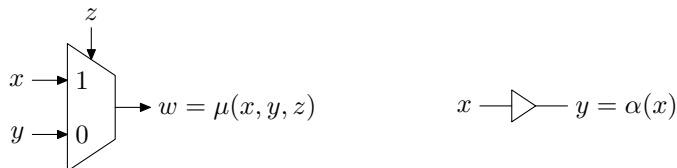


Figure 1: Basic building blocks: multiplexer (μ) and affine transformation (α).

Motivation An affine multiplexing network’s motivating capabilities are ultimately realized by its encoding in linear arithmetic. This encoding allows one to ask and answer quantitative questions about the network function, or any other function when expressed as an AMN, using Satisfiability Modulo Theory (SMT) or Mixed Integer Programming (MIP) solvers.

This formal encoding feature of AMNs makes them particularly attractive for the analysis and verification of *control systems* with nonlinear components in the loop, including switched systems, classifiers, and neural networks. In the first part of this paper, §1, we formally define and explore examples of AMNs, and explain how to encode them using SMT and MIP. Ultimately motivated by the modeling capabilities of AMNs in the loop with control systems, we describe a powerful counterexample-guided computational procedure to search for Lyapunov functions, which we will describe in §2. We give several extended examples in §3, and conclude with a number of directions for future work in §4.

1.2 Formal definition

Definition 1 (Affine Multiplexing Network). An *affine multiplexing network* is a real vector function $\varphi : \mathbf{R}^q \rightarrow \mathbf{R}^p$ that can be expressed recursively as

$$\varphi(x) ::= x \mid \alpha(\varphi_1(x)) \mid \mu(\varphi_1(x), \varphi_2(x), \varphi_3(x)), \quad (3)$$

where x is the (vector) input variable, μ is the multiplexing function (1), α is any affine transformation of the form (2), and $\varphi_1, \dots, \varphi_3$ are any other affine multiplexing networks with compatible dimensions.

We distinguish between the network function φ (or $\varphi[x]$, where the input variable x has not been bound to any particular value), and its evaluation at an input, $\varphi(a) = \varphi[x := a]$. For a given assignment to the input variable x , each terminal expression in (3) evaluates to a vector of appropriate dimension. The entire network φ is parameterized by the weights and biases of all its affine expressions, which we lump into a single r -dimensional vector $\theta \in \mathbf{R}^r$. When important, we write $\varphi_\theta(x)$ to stress that the function φ is parameterized by θ .

The input and output dimensions q and p can be arbitrary and different for different AMN instances, as long as the full recursive expression (3) makes sense. Constants are affine transformations independent of the input, e.g., $\alpha(x) = c$. As seen in the next section, many common functions can be rewritten in AMN form.

1.3 Examples

Example 1 (Maximum). The function $\max : \mathbf{R}^2 \rightarrow \mathbf{R}$ ($q = 2, p = 1$) that computes the maximum of two numbers can be expressed as an AMN. Let $x = (x_1, x_2) \in \mathbf{R}^2$ be the input variable, and define affine transformations $\alpha_i :$

$\mathbf{R}^2 \rightarrow \mathbf{R}$, $i = 1, \dots, 3$, by

$$\alpha_1(x_1, x_2) = x_1, \quad \alpha_2(x_1, x_2) = x_2, \quad \text{and} \quad \alpha_3(x_1, x_2) = -x_1 + x_2.$$

By composing α_1 , α_2 , and α_3 with a multiplexer, we can define the network

$$\varphi^{\max}(x_1, x_2) = \mu(\alpha_1(x_1, x_2), \alpha_2(x_1, x_2), \alpha_3(x_1, x_2)). \quad (4)$$

It follows that $\varphi^{\max}(x_1, x_2)$ evaluates to $\max(x_1, x_2)$, for all $x \in \mathbf{R}^2$. We often suppress the affine transformations α_i for notational convenience, and record the expression (4) directly as $\varphi^{\max}(x_1, x_2) = \mu(x_1, x_2, -x_1 + x_2)$, see Figure 2a. \square

Example 2 (Rectification). A common activation nonlinearity in neural networks is the rectifier (also known as the ramp, or a rectified linear unit (ReLU)) function $r : \mathbf{R} \rightarrow \mathbf{R}$ ($q = p = 1$), where $r(x) = \max(x, 0)$. Using Example 1 with $x_2 = 0$ (constant) gives the AMN $\varphi^r(x) = \mu(x, 0, -x)$, see Figure 2b. \square

Example 3 (Saturation). The saturation function $\text{sat} : \mathbf{R} \rightarrow \mathbf{R}$ ($q = p = 1$), defined as

$$\text{sat}(x) = \begin{cases} -1, & \text{if } x \leq -1, \\ x, & \text{if } -1 < x < 1, \\ 1, & \text{otherwise,} \end{cases}$$

can be written as $\varphi^{\text{sat}}(x) = \mu(1, \mu(-1, x, x + 1), -x + 1)$, see Figure 2c. \square

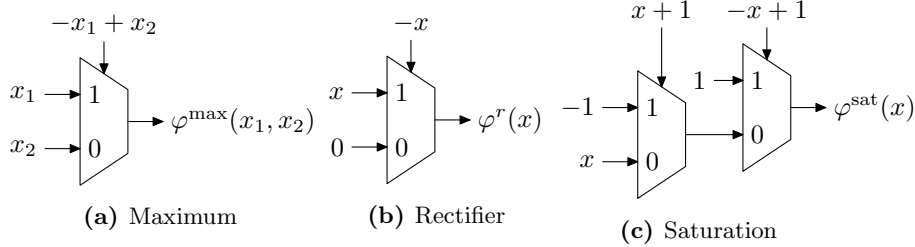


Figure 2: Selected piecewise affine functions.

Example 4 (Smooth activations). Functions that are not piecewise affine, like $f(x) = x^2$ (square), $f(x) = (1 + e^{-x})^{-1}$ (sigmoid), and $f(x) = \tanh(x)$, cannot be represented exactly as an AMN. They can, however, be approximated arbitrarily well by an AMN on any compact interval. \square

We now introduce a specific network as a running example to demonstrate encoding and training of AMNs.

Example 5 (Triplexer). The AMN illustrated in Figure 3 is meant to resemble a classical single-input/single-output, two-layer, feedforward networks. It uses four multiplexers, arranged in a feedforward topology, with two affine layers and two nonlinear layers.

This triplexer AMN $\varphi_{\theta}^{\text{TRI}} : \mathbf{R} \rightarrow \mathbf{R}$ ($q = p = 1$) can be used to approximate a real-valued function. It is parameterized by the 24 weights and biases $\theta = (a_1, b_1, \dots, f_4) \in \mathbf{R}^{24}$ making up 12 affine transformations:

$$\begin{aligned}
 \text{First layer weights:} & \quad \left\{ \begin{array}{l} x_1 := a_1x + b_1 \\ y_1 := c_1x + d_1 \\ z_1 := e_1x + f_1 \end{array} \right. \quad \cdots \quad \left\{ \begin{array}{l} x_3 := a_3x + b_3 \\ y_3 := c_3x + d_3 \\ z_3 := e_3x + f_3 \end{array} \right. \\
 \text{First nonlinearity:} & \quad \left\{ \begin{array}{l} w_1 := \mu(x_1, y_1, z_1) \\ w_2 := \mu(x_2, y_2, z_2) \\ w_3 := \mu(x_3, y_3, z_3) \end{array} \right. \\
 \text{Second layer weights:} & \quad \left\{ \begin{array}{l} x_4 := a_4w_2 + b_4 \\ y_4 := c_4w_3 + d_4 \\ z_4 := e_4w_1 + f_4 \end{array} \right. \\
 \text{Second nonlinearity:} & \quad \left\{ \begin{array}{l} y := \mu(x_4, y_4, z_4) \end{array} \right. \quad \square
 \end{aligned}$$

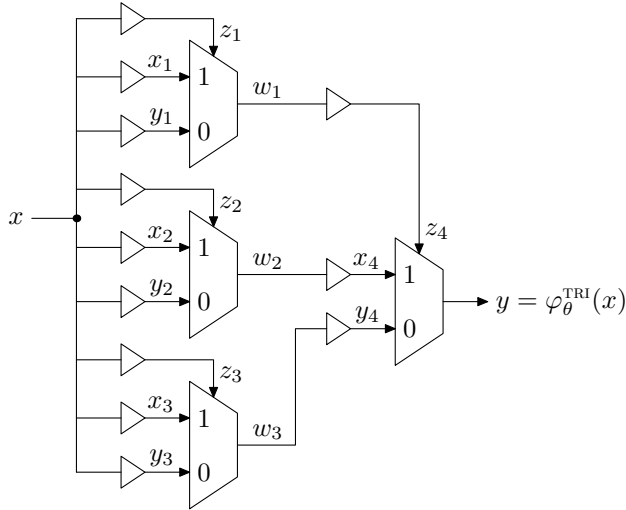


Figure 3: The “triplexer,” a 2-layer 4-mux affine multiplexing network.

Discontinuous functions Following the previous examples, it is possible to express any continuous, piecewise affine function exactly as an AMN. However, because the multiplexing function μ is effectively an if-then-else statement, we can express many discontinuous functions as AMNs as well, making these networks strictly more powerful for modeling switched and hybrid systems than

neural networks with continuous nonlinearities. The next example illustrates the powerful modeling capability of AMNs in dynamical systems.

Example 6 (Switched system). The dynamical system with state dependent switching

$$x(t+1) = \begin{cases} A^-x(t), & \text{if } x_1(t) \leq 0, \\ A^+x(t), & \text{otherwise,} \end{cases} \quad t = 0, 1, 2 \dots, \quad (5)$$

where $A^-, A^+ \in \mathbf{R}^{n \times n}$ are given dynamics matrices, and $x(t) \in \mathbf{R}^n$ is the state at time t , is equivalent to the dynamical system $x(t+1) = \varphi^{\text{sw}}(x(t))$. The state transition function $\varphi^{\text{sw}} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined by the affine multiplexing network

$$\varphi^{\text{sw}}(x) = \mu(A^-x, A^+x, e_1^T x),$$

where $e_1 = (1, 0, \dots, 0)$ is the standard basis vector in \mathbf{R}^n . \square

Note that the (nonlinear) state transition function φ^{sw} in Example 6 need not be continuous on the switching boundary $\{0\} \times \mathbf{R}^{n-1}$; this discontinuity is in general impossible to model exactly using a neural network with continuous (e.g., sigmoid, ReLU) nonlinearities, but poses no difficulty for the AMN model, because the multiplexing function μ (eq. (1)) by design can be discontinuous in its first two arguments.

Useful AMNs Some common functions and their implementations appear in Table 1. Note that a valid AMN need not necessarily be convex, differentiable, or even continuous, as in the case of the cardinality function $\mathbf{card}(x)$. However an AMN must ultimately be expressible as a composition of multiplexers and affine transformations.

Name	Function	AMN Expression
maximum	$\max(x, y)$	$\mu(x, y, -x + y)$
minimum	$\min(x, y)$	$\mu(y, x, -x + y)$
rectification	$r(x) = \max(x, 0)$	$\mu(x, 0, -x)$
abs. value	$ x $	$\mu(-x, x, x)$
saturation	$\text{sat}(x)$	$\mu(1, \mu(-1, x, x + 1), -x + 1)$
deadzone	$\text{dz}(x)$	$\mu(x + 1, \mu(x - 1, 0, x + 1), -x + 1)$
$\ x\ _\infty$	$\max(x_1 , \max(x_2 , \dots))$	$\mu(x_1 , \mu(x_2 \dots), - x_1 + \mu(x_2 \dots))$
$\ x\ _1$	$ x_1 + \dots + x_n $	$\sum_{i=1}^n \mu(-x_i, x_i, x_i)$
$\mathbf{card}(x)$	$ \{1 \leq i \leq n \mid x_i \neq 0\} $	$\sum_{i=1}^n \mu(\mu(1, 0, x_i), 0, -x_i)$

Table 1: Implementation of common functions as affine multiplexing networks.

Note that by definition of an AMN, the enable condition for any multiplexer is $z \leq 0$. By composing multiplexers appropriately, it is possible for the enable condition to have another real comparison, e.g., $\leq, <, \geq, >, =, \neq$. For example,

the enable condition $z \geq 0$ can be emulated by negating the enable input of a multiplexing unit, $\mu(x, y, -z)$. See Table 2 for an AMN implementation of allowable comparisons.

Multiple comparisons can be composed by gating operations (AND, OR, XOR, etc.). For example, the AMN $\varphi^\wedge(x, y, z_1, z_2)$ corresponds to the AND operation,

$$\varphi^\wedge(x, y, z_1, z_2) = \begin{cases} x, & \text{if } z_1 \leq 0 \text{ and } z_2 \leq 0, \\ y, & \text{otherwise.} \end{cases}$$

Refer to Table 2 and Figure 4 for a visualization of a selected subset of these.

Gate	Definition	Expression
AND	$\varphi^\wedge(x, y, z_1, z_2)$	$\mu(\mu(x, y, z_1), y, z_2)$
OR	$\varphi^\vee(x, y, z_1, z_2)$	$\mu(x, \mu(x, y, z_1), z_2)$
NOT	$\varphi^\neg(x, y, z)$	$\mu(y, x, z)$
XOR	$\varphi^\oplus(x, y, z_1, z_2)$	$\mu(\mu(y, x, z_1), \mu(x, y, z_1), z_2)$
LE	$\varphi^\leq(x, y, z)$	$\mu(x, y, z)$
GE	$\varphi^\geq(x, y, z)$	$\mu(x, y, -z)$
LT	$\varphi^<(x, y, z)$	$\varphi^\neg(x, y, -z)$
GT	$\varphi^>(x, y, z)$	$\varphi^\neg(x, y, z)$
EQ	$\varphi^=(x, y, z)$	$\varphi^\wedge(x, y, z, -z)$
NEQ	$\varphi^\neq(x, y, z)$	$\varphi^\wedge(y, x, z, -z)$

Table 2: Implementation of various gates/comparisons as AMNs.

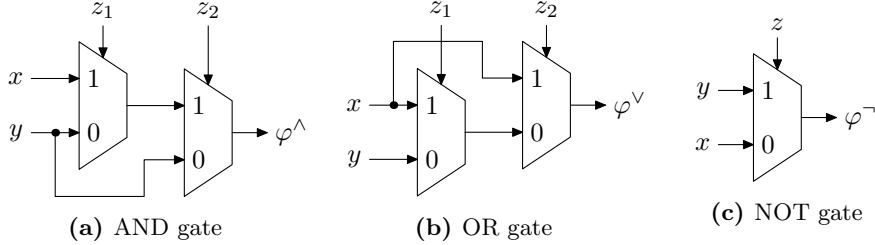


Figure 4: Selected gate functions.

1.4 Key properties

Well-definedness Not all compatibly dimensioned interconnections of multiplexers and affine transformations result in an AMN that is a well-defined function. A key requirement on AMNs is a lack of variable dependence cycles. Intuitively, phenomena such as race conditions cannot be resolved in a pure mathematical structure like an AMN—if an output of a component feeds back

to the input, then the output may not be uniquely determined by a given input. We use the concept of a computation graph to make this concept formal.

Definition 2 (Variables, direct dependency, computation graph, well-definedness). Given an AMN φ , the set $\text{Var}(\varphi)$ of internal signal *variables* is defined recursively as

$$\text{Var}(\varphi) = \begin{cases} \{x\}, & \text{if } \varphi(x) = x, \\ \{\varphi\} \cup \text{Var}(\varphi_1), & \text{if } \varphi(x) = \alpha(\varphi_1(x)), \\ \{\varphi\} \cup \text{Var}(\varphi_1) \cup \text{Var}(\varphi_2) \\ \quad \cup \text{Var}(\varphi_3), & \text{if } \varphi(x) = \mu(\varphi_1(x), \varphi_2(x), \varphi_3(x)). \end{cases}$$

For two variables $v_i, v_j \in \text{Var}(\varphi)$, we say that v_j *directly depends* on v_i if $v_j = \alpha(v_i)$ for some α , or $v_j = \mu(v_k, v_l, v_m)$, with $i \in \{k, l, m\}$. The *computation graph* of φ is a directed graph $G(\varphi) = (V, E)$, with vertices $V = \text{Var}(\varphi)$, and an edge for every direct dependency, $E = \{(v_i, v_j) \in V \times V \mid v_j \text{ directly depends on } v_i\}$. The network φ is *well-defined* if $G(\varphi)$ has no directed cycles.

Definition 3 (Inputs and outputs). Given a well-defined AMN φ , and its computation graph $G(\varphi) = (V, E)$, the sets of *input* and *output* variables are, respectively

$$\begin{aligned} \text{In}(\varphi) &= \{v \in V \mid v \text{ has no incoming edges}\}, \\ \text{Out}(\varphi) &= \{w \in V \mid w \text{ has no outgoing edges}\}. \end{aligned}$$

The computation graph encodes dependency relationships between the internal variables. The set of variables $\text{Var}(\varphi)$ contains unique names for the outputs of the constituent units of φ . Following the electronic systems metaphor, the variables are the *signals* or *wires* in a circuit diagram like Figures 1–3; variables also correspond to the *vertices* or *nodes* of $G(\varphi)$. Similarly, the computational operations μ and α correspond to *edges* in $G(\varphi)$. The nodes of $G(\varphi)$ with no incoming edges are inputs to φ (or constants), and nodes with no outgoing edges are the outputs of φ . For a given assignment to the inputs, there is a unique assignment to all internal nodes and the outputs, provided the network φ is well-defined.

The computation graph $G(\varphi_\theta^{\text{TRI}})$ for the triplexer from Example 5 is shown in Figure 5. It depicts a natural flow of information for computing a real output $y = \varphi_\theta^{\text{TRI}}(x)$ for a given real input x . In this case, the number of variables is $|V| = 17$ with $V = \{x, z_1, x_1, y_1, \dots, y\}$, and $|E| = 24$ with $E = \{(x, z_1), (x, x_1), (x, y_1), \dots, (y_4, y)\}$. Since there are no directed cycles in $G(\varphi_\theta^{\text{TRI}})$, the expression $\varphi_\theta^{\text{TRI}}$ is a well-defined function from \mathbf{R} to \mathbf{R} .

Non-uniqueness There is usually more than one way to express a given piecewise affine function as an AMN. For example, using the identity $\text{sat}(x) =$

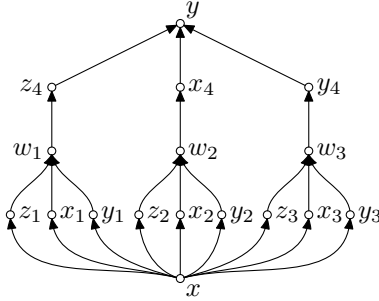


Figure 5: Computation graph $G(\varphi_{\theta}^{\text{TRI}})$ for the triplexer.

$r(x+1) - r(x-1) - 1$, we obtain an alternative expression

$$\begin{aligned} \varphi^{\text{sat}'}(x) &= \varphi^r(x+1) - \varphi^r(x-1) - 1 \\ &= \mu(x+1, 0, -x-1) - \mu(x-1, 0, -x+1) - 1, \end{aligned}$$

which is distinct from, and has a different encoding than the expression for φ^{sat} in Example 3. Nevertheless the two expressions evaluate to the same output, i.e., $\varphi^{\text{sat}}(x) = \varphi^{\text{sat}'}(x)$, for all $x \in \mathbf{R}$. This non-uniqueness means that one implementation of a given function as an AMN can be more efficient than another if that implementation uses fewer multiplexers or affine transformations, or if those multiplexers and affine transformations have a smaller dimensionality.

Universal approximation Like a classical neural network, an AMN can approximate an arbitrary nonlinear function, but unlike classical neural networks, the AMN output is allowed to be discontinuous. AMNs inherit the universal function approximation property from the neural networks they embed (cf. [Cyb89]).

1.5 Encoding

By focusing on affine transformations and affine enable conditions, we constrain the network input-output relationship to be a conjunction or disjunction of statements over the linear real arithmetic. If the inputs and outputs are unbound or partially bound, a satisfying assignment can be obtained by solving a sequence of linear programs (LPs). To see how this works, we first define a recursive procedure for converting an affine multiplexing network φ into statements over linear real arithmetic (SMT encoding), or a set of linear constraints over real and binary variables (MIP encoding).

SMT encoding For a given AMN φ with input $x \in \mathbf{R}^q$ and output $y \in \mathbf{R}^p$, the formula $\text{SMT}_\varphi[x, y]$ is a first-order logic formula given recursively as

$$\text{SMT}_x[x, y] \equiv \{y = x\}, \quad (6)$$

$$\text{SMT}_{\alpha(\varphi_1)}[x, y] \equiv \left\{ \begin{array}{l} \exists v. (y = Av + b) \\ \wedge \text{SMT}_{\varphi_1}[x, v] \end{array} \right\}, \text{ where } \alpha(\xi) = A\xi + b, \quad (7)$$

$$\text{SMT}_{\mu(\varphi_1, \varphi_2, \varphi_3)}[x, y] \equiv \left\{ \begin{array}{l} \exists u, v, w. ((w \leq 0) \rightarrow (y = u)) \\ \wedge (\neg(w \leq 0) \rightarrow (y = v)) \\ \wedge \text{SMT}_{\varphi_1}[x, u] \\ \wedge \text{SMT}_{\varphi_2}[x, v] \\ \wedge \text{SMT}_{\varphi_3}[x, w] \end{array} \right\}. \quad (8)$$

Example 7 (Triplexer (SMT)). After simplification and variable renaming, we obtain a linear real arithmetic encoding of the triplexer from from Example 5:

$$\text{SMT}_{\varphi_{\text{tri}}}^{\text{ra}}[x, y] \equiv \left\{ \begin{array}{l} \exists (x_1, y_1, z_1, \dots, x_4, y_4, z_4, w_1, w_2, w_3) \in \mathbf{R}^{15}. \\ \bigwedge_{i=1}^3 (x_i = a_i x + b_i \wedge y_i = c_i x + d_i \wedge z_i = e_i x + f_i) \\ \wedge \bigwedge_{j=1}^3 ((z_j \leq 0) \rightarrow (w_j = x_j)) \wedge (\neg(z_j \leq 0) \rightarrow (w_j = y_j)) \\ \wedge (x_4 = a_4 w_2 + b_4 \wedge y_4 = c_4 w_3 + d_4 \wedge z_4 = e_4 w_1 + f_4) \\ \wedge ((z_4 \leq 0) \rightarrow (y = x_4)) \wedge (\neg(z_4 \leq 0) \rightarrow (y = y_4)) \end{array} \right\}.$$

Note that the only unbound variables in $\text{SMT}_{\varphi_{\text{tri}}}^{\text{ra}}[x, y]$ are the input x and output y . Furthermore, every clause is an affine equation, inequality, or the logical negation of an affine equation or inequality. \square

MIP encoding Given an AMN φ , and its computation graph $G(\varphi) = (V, E)$, we define a collection of mixed integer constraints, parameterized by $x \in \mathbf{R}^q$ and $y \in \mathbf{R}^p$, over the variables $\text{Var}(\varphi)$ and additional binary variables as follows:

1. For $v \in \text{In}(\varphi)$, add the constraint $x = v$; for $w \in \text{Out}(\varphi)$, add $y = w$.
2. For each $(v_i, v_j) \in E$ with $v_j = \alpha(v_i)$, $\alpha(\xi) = A\xi + b$, add the constraint $v_j = Av_i + b$ with real (vector) variables v_i and v_j .
3. For each $(v_k, v_j), (v_l, v_j), (v_m, v_j) \in E$ with $v_j = \mu(v_k, v_l, v_m)$, add the mixed integer “big- M ” constraints

$$\begin{aligned} -Mb_j &< v_m \leq M(1 - b_j), \\ -\mathbf{1}Mb_j &\preceq v_j - v_l \preceq \mathbf{1}Mb_j, \\ -\mathbf{1}M(1 - b_j) &\preceq v_j - v_k \preceq \mathbf{1}M(1 - b_j), \quad b_j \in \{0, 1\}, \end{aligned} \quad (9)$$

with real (vector) variables v_j, v_k, v_l, v_m and binary variables $b_j \in \{0, 1\}$.

The formula $\text{MIP}_\varphi[x, y]$ is the conjunction of the constraints obtained through these steps. Every affine unit in φ corresponds to an affine equality constraint, and every multiplexer corresponds to a binary variable b_j that is true ($b_j = 1$) if and only if the corresponding enable condition is met ($v_m \leq 0$). We use the “big- M ” constraints (9), which are equivalent to the constraint $v_j = \mu(v_k, v_l, v_m)$, provided M is a large enough constant, see [Gro02].

1.6 Optimization

Recall that equality constraints like $y = h(x)$, with variables x and y , can be efficiently imposed in linear programs, and in general, in convex optimization programs (see, e.g. [BV04]). The aim of encoding an AMN in SMT or MIP is to represent constraints like $y = \varphi(x)$, with x and y as variables, and φ is an arbitrary, possibly non-affine AMN, in an optimization problem. Ultimately, we would like to be able to solve optimization programs in the form

$$\begin{aligned} & \text{minimize} && \varphi_0(x) \\ & \text{subject to} && \varphi_i(x) \leq 0, \quad i = 1, \dots, m_1, \\ & && \psi_j(x) = 0, \quad j = 1, \dots, m_2, \end{aligned} \tag{10}$$

over a variable $x \in \mathbf{R}^q$, where $\varphi_0, \dots, \varphi_{m_1}, \psi_1, \dots, \psi_{m_2}$ are arbitrary AMNs. The idea is made more clear by the graph of an AMN.

Definition 4 (Graph). The *graph* of a function $f : \mathbf{R}^q \rightarrow \mathbf{R}^p$ is the set of input-output pairs $\llbracket f \rrbracket = \{(x, y) \in \mathbf{R}^q \times \mathbf{R}^p \mid y = f(x)\}$. For a first-order logic formula $\psi[x, y]$ with free variables x, y , it is the set of satisfying assignments, $\llbracket \psi \rrbracket = \{(c_1, c_2) \in \mathbf{R}^q \times \mathbf{R}^p \mid \psi[x := c_1, y := c_2]\}$.

Example 8. For the real-valued function $f(x) = x^2$, and the first-order logic formula $\psi[x, y] \equiv \exists z. (x \geq z) \wedge (y \geq 0)$, we have $\llbracket f \rrbracket = \llbracket \psi \rrbracket = \mathbf{R} \times \mathbf{R}_+$.

Theorem 1 (Encoding). *Given a well-defined AMN φ ,*

$$\llbracket \varphi \rrbracket = \llbracket \text{SMT}_\varphi[x, y] \rrbracket = \llbracket \text{MIP}_\varphi[x, y] \rrbracket.$$

Proof. By construction. □

Querying the solver As a consequence of Theorem 1, we can represent the graph $\llbracket \varphi \rrbracket$ of an AMN φ as a conjunction or disjunction of linear atoms. We can formulate nonconvex feasibility and optimization problems over real vector variables, and solve them using an SMT or MIP solver.

For example, the (nonconvex) feasibility problem

$$\begin{aligned} & \text{find} && (x, y) \in \mathbf{R}^{q \times p} \\ & \text{subject to} && y = \varphi(x) \end{aligned} \tag{11}$$

has a solution if and only if there exists $(x, y) \in \llbracket \varphi \rrbracket$. In other words, the problem (11) is equivalent to the problem

$$\begin{aligned} & \text{find} && (x, y) \in \mathbf{R}^{q \times p} \\ & \text{subject to} && (x, y) \in \llbracket \varphi \rrbracket, \end{aligned}$$

which can be found by posing the query

$$\exists x \in \mathbf{R}^q . \exists y \in \mathbf{R}^p . \text{SMT}_\varphi[x, y] \tag{12}$$

to an SMT solver, or replacing the constraint $y = \varphi(x)$ with $\text{MIP}_\varphi[x, y]$ in a MIP solver. Moreover, the procedure for translating the feasibility problem (11) into the formal problem (12) suitable for an SMT solver using the linear theory is entirely constructive and mechanical, following the steps outlined in §1.5, and complete in the sense that the problem (11) is feasible if and only if (12) is SAT. Conversely, the problem (11) is infeasible if and only if the query (12) is UNSAT.

Constrained optimization using bisection Constrained optimization involving AMNs can be accomplished by a sequence of feasibility queries to an SMT or MIP solver. For example, consider the simplified (nonconvex) optimization problem

$$\begin{aligned} & \text{minimize} && \varphi_0(x) \\ & \text{subject to} && \varphi_i(x) \leq 0, \quad i = 1, \dots, m \end{aligned} \tag{13}$$

over a variable $x \in \mathbf{R}^q$, where $\varphi_i : \mathbf{R}^q \rightarrow \mathbf{R}$, $i = 0, \dots, m$ are AMNs. The optimal objective value satisfies $\varphi_0^* \leq t$ if and only if the SMT query

$$\exists x \in \mathbf{R}^q . \exists y_0, \dots, y_m \in \mathbf{R} . \bigwedge_{i=0}^m \text{SMT}_{\varphi_i}[x, y_i] \wedge \bigwedge_{i=1}^m (y_i \leq 0) \wedge (y_0 \leq t) \tag{14}$$

is SAT. We can minimize the function $\varphi_0(x)$ by bisection on t through a sequence of feasibility calls of the form (14), see [BV04, §4.2.5]. If the initial interval $[l, u]$ contains φ_0^* , then the number of SMT calls needed to compute φ_0^* to tolerance $\epsilon > 0$ using the bisection method is at most $\lceil \log_2((u - l)/\epsilon) \rceil$. Moreover, an ϵ -suboptimal value x_ϵ^* with $|\varphi_0(x_\epsilon^*) - \varphi_0^*| \leq \epsilon$ is obtained directly from the last query (14) that returned SAT.

AMNET modeling toolbox The bisection procedure is implemented in our open-source modeling package, AMNET¹. In addition to allowing one to define and evaluate AMNs with the building blocks μ and α , AMNET allows one to define new AMNs by composing existing AMNs in a disciplined manner, automatically convert neural networks from to AMNs, and solve optimization problems with AMN objectives and constraints. See §3 for example applications.

¹<https://github.com/ipapusha/amnet>

1.7 Training

1.7.1 Gradient descent

Finding weights of an AMN to solve a regression or classification problem can be accomplished with a modified version of the gradient descent algorithm. In such problems the goal is to find weights θ to minimize an objective $J(\theta)$; example objectives could be least squares or negative log-likelihood.

For illustration, consider a “perceptron” network $\varphi_\theta^{\text{PER}} : \mathbf{R}^n \rightarrow \mathbf{R}$ consisting of a single affine layer with a multiplexing nonlinearity,

$$\varphi_\theta^{\text{PER}}(x) = \mu(\alpha(x), \beta(x), \gamma(x)), \quad \begin{cases} \alpha(x) = a^T x + b, \\ \beta(x) = c^T x + d, \\ \gamma(x) = e^T x + f, \end{cases}$$

where $\theta = (a, b, c, d, e, f) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}$ are the weights parameterizing the network.

To apply gradient descent, it is necessary to compute a descent direction $\nabla_\theta J(\theta)$, which requires computing the gradient of the AMN with respect to its weights. However, the function $\varphi_\theta^{\text{PER}}(x)$ is not necessarily differentiable in θ . It is differentiable almost everywhere, except on the set $\{\theta \mid e^T x + f = 0\}$ having measure zero, because the multiplexing nonlinearity μ is not necessarily continuous there. However the (weak) derivative of the nonlinearity μ can be written in terms of the nonlinearity μ itself,

$$\begin{aligned} \nabla_\theta(\varphi_\theta^{\text{PER}}(x)) &= (\nabla_a(\varphi_\theta^{\text{PER}}(x)), \nabla_b(\varphi_\theta^{\text{PER}}(x)), \dots, \nabla_f(\varphi_\theta^{\text{PER}}(x))) \\ &= \begin{bmatrix} \mu(x, 0, \gamma(x)) \\ \mu(1, 0, \gamma(x)) \\ \mu(0, x, \gamma(x)) \\ \mu(0, 1, \gamma(x)) \\ 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{15}$$

allowing us to define a version of backpropagation where the nondifferentiable weights do not change.

algorithm: Gradient descent for AMNs

given: an AMN, training objective $J(\theta)$, $k = 0$, initial $\theta^{(0)}$, learning rates α_k , tolerance $\epsilon > 0$

repeat:

1. $\theta^{(k+1)} := \theta^{(k)} - \alpha_k \nabla_\theta(J(\theta^{(k)}))$
2. $k := k + 1$

until: $\|\nabla_\theta(J(\theta^{(k)}))\| \leq \epsilon$ (or another stopping criterion)

Performance modifications to the gradient descent algorithm, including momentum, batching, and dropout, are ready extensions. One deficiency of gradient descent stems from the enable parameters (e and f in φ^{PER} above) not changing with training. The network is stuck with the initial weights and biases parameterizing the enable components of θ , because those derivatives are set to zero, see eq. (15). However, an AMN trained by backpropagation in our experiments can still be remarkably expressive if the stuck parameters are initially well-dispersed.

In the next section, we will use the SMT encoding to suggest a novel—though much less efficient—algorithm that trains all AMN parameters at the same time without stuck enable weights.

1.7.2 SMT embedding in NL theory

Definition 5. (Dual of an AMN) Given an AMN $\varphi_\theta : \mathbf{R}^q \rightarrow \mathbf{R}^p$, where $\theta \in \mathbf{R}^r$ are the parameters, i.e., (stacked) weights and biases of all the affine units, its dual is a function $\varphi^\circ : \mathbf{R}^r \rightarrow \mathbf{R}^p$ such that $\varphi_x^\circ(\theta) = \varphi_\theta(x)$.

In other words, the dual φ° is the same as the original AMN φ with the roles of the input variable x and parameters θ reversed. An encoding of φ° can be obtained from the SMT encoding $\text{SMT}_\varphi[x, y]$ by adding an existential quantifier for every component of θ and assigning x . We are being intentionally agnostic about the stacking and ordering of the weights and biases of a network φ into a parameter vector θ , because there can be many consistent ways to do it.

Example 9 (Dual of perceptron). The dual of the single-layer “perceptron” network

$$\varphi_\theta(x) = \mu(\alpha(x), \beta(x), \gamma(x)), \quad \begin{cases} \alpha(x) = a^T x + b, \\ \beta(x) = c^T x + d, \\ \gamma(x) = e^T x + f, \end{cases}$$

where $\theta = (a, b, c, d, e, f) \in \mathbf{R}^n \times \mathbf{R} \times \cdots \times \mathbf{R}^n \times \mathbf{R}$ are the weights parameterizing the network, is given by

$$\varphi_x^\circ(\theta) = \mu(\alpha_x^\circ(\theta), \beta_x^\circ(\theta), \gamma_x^\circ(\theta)), \quad \begin{cases} \alpha_x^\circ(\theta) = x^T a + b, \\ \beta_x^\circ(\theta) = x^T c + d, \\ \gamma_x^\circ(\theta) = x^T e + f, \end{cases}$$

where x is the (fixed) parameter and $\theta = (a, b, c, d, e, f)$ is the variable. \square

The dual φ° in the previous example is itself an AMN, but this need not be the case in general. As a result, the relation between the input and output of φ° cannot always be encoded in SMT using a linear theory. For example, by interchanging the roles of the parameters and the input, the dual of the triplexer (Example 5) involves products between existentially quantified variables, e.g., a_4 and w_2 .

Thus, by following the SMT encoding procedure with the nonlinear theory (NL), we can still find elements of $\llbracket \varphi^\circ \rrbracket$ in a decidable way. However, working

with the nonlinear theory is much less computationally efficient than working with the linear theory.

We use the concept of a dual AMN to propose a training procedure for the weights and biases by encoding them as variables in an SMT query, and using consistency training.

Consistency training Let $\mathcal{D} = \{(x^{(i)}, y^{(i)})\}_{i=1}^N$ be a data set of ordered pairs of training data points. Given an AMN φ_θ , we say that the network is ϵ -consistent with the data point $(x, y) \in \mathcal{D}$ if $\|y - \varphi_\theta(x)\|_1 \leq \epsilon$. The set of parameters defining all ϵ -consistent networks,

$$\Theta_\epsilon = \{\theta \in \mathbf{R}^r \mid \|y - \varphi_\theta(x)\|_1 \leq \epsilon \text{ for all } (x, y) \in \mathcal{D}\},$$

is SMT representable in the nonlinear theory. The encoding of Θ_ϵ can be derived by observing that for a fixed $\epsilon \geq 0$, the set Θ_ϵ is nonempty if and only if

$$\exists \theta \in \mathbf{R}^r . \bigwedge_{(x,y) \in \mathcal{D}} \|y - \varphi_x^\circ(\theta)\|_1 \leq \epsilon \quad (16)$$

is satisfiable. The expression (16) is SMT representable, because each conjunct is SMT representable.

There are many other ways to define consistency. A weighted sum of norms can be used to emphasize certain training samples. Each conjunct involves a 1-norm here, but it is possible to use, e.g., a weighted norm, a ramp function (regret), or any other SMT representable function. By restricting to general p -norms, all constraints remain polynomial equations and inequalities.

The consistency viewpoint is useful because we can *train* or *retrain* a network φ_θ by posing the query (16) to an SMT solver. If the result is SAT, then the parameter $\theta_0 \in \Theta_\epsilon$ obtained from the query defines a network φ_{θ_0} with which every point in \mathcal{D} is ϵ -consistent. If the result is UNSAT, then no assignment to the network parameters can result in an ϵ -consistent network for the whole data set. In such a case, we can either remove offending examples from \mathcal{D} , increase ϵ , or make the network more expressive by adding extra architectural layers. A similar data set consistency idea was used in [PWT18] to solve inverse optimal control problems with regular language specifications.

Robust consistency training Suppose we would like to train a network that is robust with respect to, e.g., bounded (rectangular) perturbations on the input. In the consistency framework, this might correspond to the query

$$\exists \theta \in \mathbf{R}^r . \forall \delta \in [\delta^-, \delta^+]^q . \bigwedge_{(x,y) \in \mathcal{D}} \|y - \varphi_{x+\delta}^\circ(\theta)\|_1 \leq \epsilon. \quad (17)$$

The intuition is this: since $(x + \delta)$ and θ multiply together in the encoding of (17), and the query (17) is in *exists-forall* (EF) form, we can synthesize a robust ϵ -consistent network by a sequence of queries to an SMT solver; see, e.g. [CSRB13].

2 AMNs in the loop

2.1 Autonomous stability

In this section, we are concerned with defining a procedure that formally and automatically proves properties of the autonomous discrete time nonlinear system

$$x(t+1) = \varphi(x(t)), \quad x(0) = x_0, \quad t = 0, 1, 2, \dots, \quad (18)$$

where $x(t) \in \mathbf{R}^n$ is the state at time t , $x_0 \in \mathcal{X}_0 \subseteq \mathbf{R}^n$ is the initial condition, \mathcal{X}_0 is the initial set, and $\varphi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is any well-defined AMN.

Example 10 (Neural networks in the loop). A known, memoryless state-feedback neural network controller $u(t) = \text{NN}(x(t))$, $\text{NN} : \mathbf{R}^n \rightarrow \mathbf{R}^m$, which uses only piecewise affine nonlinearities (e.g., ReLU), stabilizes the linear system

$$x(t+1) = Ax(t) + Bu(t), \quad u(t) = \text{NN}(x(t)), \quad t = 0, 1, \dots, \quad (19)$$

if and only if the system (18) is stable with $\varphi(x) = Ax + B \cdot \text{NN}(x)$. The system is illustrated in Figure 6. \square

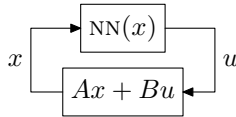


Figure 6: Neural network in the loop.

Example 11 (Variable-gain control). The phase-based variable-gain nonlinearity from [HWN16, eq. (4)],

$$\varphi(e, \dot{e}) = \begin{cases} \alpha e, & \text{if } e\dot{e} > 0, \\ 0, & \text{otherwise,} \end{cases}$$

can be written as the network $\varphi^{\text{vgc}}(e, \dot{e}) = \varphi^{\vee}(\varphi^{\vee}(0, \alpha e, -e, -\dot{e}), \alpha e, e, \dot{e})$, (see Table 2). The network satisfies $\varphi^{\text{vgc}}(e, \dot{e}) = \varphi(e, \dot{e})$ for all e, \dot{e} . \square

Example 12 (Linear feedback system). With $u(t) = \text{NN}(x(t)) = Kx(t)$, where $K \in \mathbf{R}^{m \times n}$, the system (18) corresponds to a linear feedback controller. Note that we can write $\text{NN}(x(t))$ as single-layer neural network with at most $2m$ internal units and $2mn$ weights by taking advantage of the identity $x = r(x) - r(-x)$. The weights of the neural network are K and $-K$. \square

Example 13 (Saturation nonlinearity). The single-input system [Joh03, ex 2.1]

$$x(t+1) = Ax(t) + b \cdot \text{sat}(v(t)), \quad v(t) = k^T x(t), \quad t = 0, 1, \dots$$

with saturation, where the saturation function is

$$\text{sat}(v) = \begin{cases} -1, & v \leq -1 \\ v, & -1 < v < 1 \\ 1, & v \geq 1, \end{cases}$$

can be written as a linear system with neural network feedback. The saturation function is an affine combination of ReLU nonlinearities, $\text{sat}(x) = r(x + 1) - r(x - 1) - 1$. In this case, the neural network function is given by $\text{NN} : \mathbf{R}^n \rightarrow \mathbf{R}$,

$$\text{NN}(x(t)) = \text{sat}(k^T x(t)) = r(k^T x(t) + 1) - r(k^T x(t) - 1) - 1.$$

The controller weights $k \in \mathbf{R}^m$ are encoded as weights in the neural network. \square

2.2 Counterexample-guided Lyapunov search

To prove autonomous stability for systems like ones in the previous section §2.1, we will synthesize a Lyapunov function, which is positive definite, and decreases along trajectories of the system (18). The following procedure searches for a Lyapunov function $V : \mathbf{R}^n \rightarrow \mathbf{R}$ from a candidate class \mathcal{V} by keeping track of a *counterexample set* $\mathcal{C} \subseteq \mathcal{X}_0$ of initial points. Early work in this direction includes [CSRB13, KDSA14, RS15].

1. *Select candidate*: Choose a candidate Lyapunov function V from \mathcal{V} that decreases on \mathcal{C} , i.e., V should satisfy

$$\forall x_0 \in \mathcal{C}. (V(0) = 0) \wedge (x_0 \neq 0 \rightarrow V(x_0) > 0) \wedge (V(\varphi(x_0)) - V(x_0) < 0).$$

If such a V cannot be found within \mathcal{V} , return UNKNOWN.

2. *Generate counterexample*: Attempt to find a point $x_c \in \mathcal{X} \setminus \mathcal{C}$ at which the candidate V fails to decrease, i.e., pose the query

$$\exists x_c \in \mathcal{X}. (x_c \neq 0 \wedge V(x_c) \leq 0) \vee (V(\varphi(x_c)) - V(x_c) \geq 0).$$

3. *Update*: If such x_c is found, update $\mathcal{C} := \mathcal{C} \cup \{x_c\}$ and go to step 1. Otherwise, return STABLE.

If the first step fails, we must terminate the procedure, and say nothing about the asymptotic stability or instability of (18), unless \mathcal{V} is known to be expressive enough, e.g., polyhedral for certain φ [Bit88]. Furthermore, the procedure need not terminate. The art and science of this counterexample-guided prescription is in choosing a computationally tractable \mathcal{V} and update procedures.

For example, let \mathcal{A}_N be the set of all well-defined AMNs with at most N multiplexers. If $\mathcal{V} \subseteq \mathcal{A}_N$, and \mathcal{X} is SMT representable, then Step 2 is equivalent to the query

$$\begin{aligned} \exists x_c \in \mathcal{X}, \exists x_c^+ \in \mathbf{R}^n, \exists v_c, v_c^+ \in \mathbf{R}. \text{SMT}_V[x_c, v_c] \wedge ((x_c \neq 0 \wedge v_c \leq 0) \\ \vee (\text{SMT}_\varphi[x_c, x_c^+] \wedge \text{SMT}_V[x_c^+, v_c^+] \wedge v_c^+ - v_c \geq 0)), \end{aligned}$$

where we used Theorem 1 to rewrite the constraint

$$x_c^+ = \varphi(x_c) \iff (x_c, x_c^+) \in \llbracket \varphi \rrbracket \iff \text{SMT}_\varphi[x_c, x_c^+],$$

and similarly,

$$v_c = V(x_c) \iff (x_c, v_c) \in \llbracket V \rrbracket \iff \text{SMT}_V[x_c, v_c].$$

General framework In general, we can often abstract questions about control systems to searches for a Lyapunov function satisfying a stability property,

$$\exists V \in \mathcal{V}. \forall x \in \mathcal{X}. \text{LYAP}(V, x), \quad (20)$$

where $\text{LYAP}(V, x)$ is a formula that includes relevant Lyapunov stability conditions. For example the formula LYAP might be:

- *Global stability:* $\mathcal{X} = \mathbf{R}^n$

$$\text{LYAP}(V, x) \equiv (V(0) = 0) \wedge (x \neq 0 \rightarrow V(x) > 0) \wedge (V(x^+) - V(x) < 0)$$

- *Region of attraction:* $\mathcal{X} \subseteq \mathbf{R}^n$

$$\text{LYAP}(V, x) \equiv (V(0) = 0) \wedge (x \neq 0 \rightarrow V(x) > 0) \wedge (V(x^+) - V(x) < 0)$$

- *Decay rate:*

$$\text{LYAP}(V, x) \equiv (V(0) = 0) \wedge (x \neq 0 \rightarrow V(x) > 0) \wedge (V(x^+) - \gamma V(x) \leq 0)$$

- *Positively invariant set:*

$$\text{LYAP}(V, x) \equiv (V(0) = 0) \wedge (x \neq 0 \rightarrow V(x) > 0) \wedge (V(x) \leq 0 \rightarrow V(x^+) \leq 0)$$

The problem (20) is in exists-forall form, and can be tackled by the general counterexample-guided procedure below, provided that the following subprocedures are tractable:

$$\text{E-SOLVE}(\mathcal{X}_c) \equiv \exists V \in \mathcal{V}. \bigwedge_{x_c \in \mathcal{X}_c} \text{LYAP}(V, x_c)$$

$$\text{F-SOLVE}(V) \equiv \exists x \in \mathcal{X}. \neg \text{LYAP}(V, x)$$

algorithm: Counterexample-guided Lyapunov search

initialize: $k := 0$, $x_0 \in \mathcal{X}$, $\mathcal{X}_0 := \{x_0\}$

repeat:

1. Search for a candidate Lyapunov function.

```

if E-SOLVE( $\mathcal{X}_k$ ) then:
     $V_k :=$  solution to E-SOLVE( $\mathcal{X}_k$ )
else return FALSE/UNKNOWN
2. Generate counterexample.
    if F-SOLVE( $V_k$ ) then:
         $x_{k+1} :=$  solution to F-SOLVE( $V_k$ )
    else return TRUE
3. Update counterexample set.
     $\mathcal{X}_{k+1} := \mathcal{X}_k \cup \{x_{k+1}\}$ 
     $k := k + 1$ 
until: stopping criterion

```

3 Extended examples

3.1 Verifying a region of attraction

For an initial application of our counterexample-guided Lyapunov function synthesis procedure, we consider the linear dynamical system $x(t+1) = Ax(t)$, where the 2×2 matrix

$$A = \begin{bmatrix} 0.7005 & -0.2638 \\ -0.2278 & -0.4627 \end{bmatrix}$$

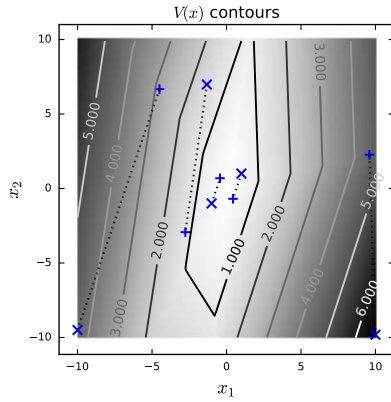
is globally (Schur) stable, with spectral radius $\rho(A) = 0.75$. The origin of the dynamical system is globally exponentially stable.

We take \mathcal{V} to be the class of max-of-affine Lyapunov functions. In general this class (or the class of AMN-representable) Lyapunov function candidates may not be large enough to prove global exponential stability, even of linear dynamical systems. So instead, we verify local stability in the box $\mathcal{B} = \{x \in \mathbf{R}^2 \mid \|x\|_\infty \leq 10\}$. In other words, we verify the weaker claim that the set \mathcal{B} is a region of attraction for the equilibrium at the origin.

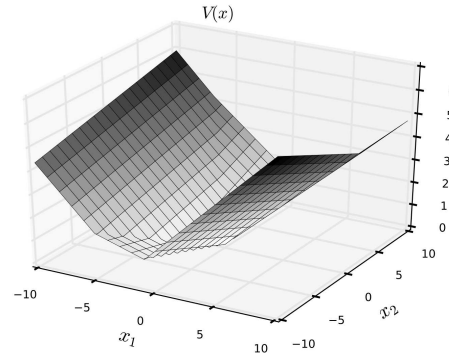
Our algorithm terminates with the piecewise affine Lyapunov function

$$V(x) = \max\{x_2, -0.1612x_1 - 0.1020x_2, 0.4614x_1 + 0.0155x_2, \\ -0.4212x_1 + 0.1433x_2, -0.5156x_1 + 0.0796x_2, 0.5036x_1 - 0.1632x_2\}$$

using automatically generated counterexamples that were all constrained to \mathcal{B} , see Figure 7. Although the function $V(x)$ is a Lyapunov function by construction, we can illustrate that it decreases along trajectories of the dynamical system $x(t+1) = Ax(t)$ by simulating the system at several initial conditions within \mathcal{B} , and tracking the value of $V(x(t))$, see Figure 8.



(a) Contours of $V(x)$



(b) 3D visualization of $V(x)$

Figure 7: Contours (left, solid lines) and 3D visualization (right) of a candidate piecewise affine function, synthesized to decrease from the initial conditions (counterexamples) marked by a cross, \times , to their state-space locations marked by a plus, $+$, at the next time step. The counterexample conditions were generated automatically using our algorithm. The resulting function $V(x)$ certifies asymptotic stability of the origin for any initial condition in the box $\mathcal{B} = \{x \in \mathbf{R}^2 \mid \|x\|_\infty \leq 10\}$.

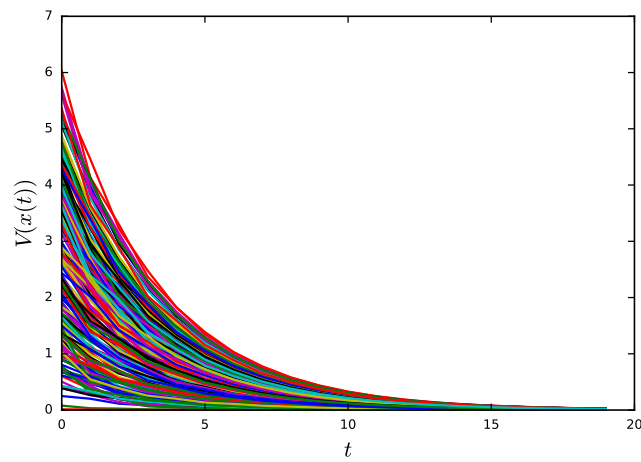


Figure 8: Lyapunov function value for trajectories starting at 200 random points in the box $\mathcal{B} = \{x \in \mathbf{R}^2 \mid \|x\|_\infty \leq 10\}$.

3.2 Verification of λ -contractive dynamical systems

In this example, we verify the λ -contractiveness of a given closed loop dynamical system, following the example in [Mil99].

Definition 6 (λ -contractive). Let G be a matrix in $\mathbf{R}^{m \times n}$ and w a vector in \mathbf{R}^m . Define the polyhedron $S(G, w) = \{x \mid Gx \preceq w\}$. The set $S(G, w)$ is λ -contractive with respect to the system (18) if there is a real $0 < \lambda < 1$ such that $x(t+1) \in S(G, w\epsilon\lambda)$ for all $0 < \epsilon \leq 1$ and all $x(t) \in S(G, w\epsilon)$, i.e.

$$\Phi : \quad \forall x. \forall \epsilon. [(0 < \epsilon \leq 1) \wedge (x \in S(G, w\epsilon))] \rightarrow (\varphi(x) \in S(G, w\epsilon\lambda)).$$

Consider the closed loop system with state-feedback control represented by the state equations

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ u(t) &= \text{sat}(Fx(t)), \end{aligned} \tag{21}$$

where $x(t) \in \mathbf{R}^n$, $u \in \mathbf{R}$, $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^n$, and

$$\text{sat}(Fx) = \begin{cases} -u^{\min} & \text{if } Fx < -u^{\min}, \\ Fx & \text{if } -u^{\min} \leq Fx \leq u^{\max}, \\ u^{\max} & \text{if } Fx > u^{\max}. \end{cases}$$

In other words, the autonomous system has $\varphi(x(t)) = Ax(t) + B \text{sat}(Fx(t))$.

Given $G \in \mathbf{R}^{r \times n}$ and $w \succ 0 \in \mathbf{R}^r$, we can test the λ -contractiveness of $S(G, w)$ by defining two AMN functions $V_1, V_2 : \mathbf{R}^n \rightarrow \mathbf{R}$, whose zero-sublevel sets represent the region inside the polyhedron $S(G, w)$. We rearrange the inequality $Gx \preceq w$, where g_i is the i -th row of G , to give

$$\begin{aligned} V_1(x) &= \max_i (g_i x - \epsilon w_i), \\ V_2(\varphi(x)) &= \max_i (g_i \varphi(x) - \epsilon \lambda w_i). \end{aligned}$$

The negative of the condition Φ is

$$\begin{aligned} \neg\Phi &= \exists x, \epsilon. x \in S(G, w\epsilon) \wedge 0 < \epsilon \leq 1 \wedge \neg(\varphi(x) \in S(G, w\epsilon\lambda)) \\ &= \exists x, \epsilon. (V_1(x) \leq 0) \wedge (V_2(x) > 0) \wedge (0 < \epsilon \leq 1) \end{aligned}$$

If there exist x and ϵ that satisfy $\neg\Phi$, then the polyhedron $S(G, w)$ is not λ -contractive with respect to system (21). However, if $\neg\Phi$ is UNSAT, then we can say that the polyhedron $S(G, w)$ is λ -contractive within the region of nonlinear behavior of system (21).

The example polyhedron $S(G, w)$ in [Mil99] can be verified as λ -contractive:

$$G = \begin{bmatrix} 0.2888 & -1.8350 \\ 0.9650 & -2.0576 \\ 1.0008 & 1.7891 \\ 1.5951 & -1.9866 \\ 2.0707 & -2.0590 \\ -1.4970 & -1.5864 \\ -0.2888 & 1.8350 \\ -0.9650 & 2.0576 \\ -1.0008 & -1.7891 \\ -1.5951 & 1.9866 \\ 1.4970 & 2.0590 \\ -2.0707 & 1.5864 \end{bmatrix}, \quad w = \begin{bmatrix} 35.4375 \\ 48.2116 \\ 48.1152 \\ 62.5184 \\ 62.3934 \\ 76.2996 \\ 35.4375 \\ 48.2116 \\ 48.1152 \\ 62.5184 \\ 62.3934 \\ 76.2996 \end{bmatrix}.$$

However, if we scale w by $\delta = 1.01$, then we can find a counterexample,

$$x(t) = \begin{bmatrix} 38.9278177 \\ 2.27698913 \end{bmatrix}, \quad x(t+1) = \begin{bmatrix} 32.28074873 \\ -5.83874012 \end{bmatrix}, \quad \epsilon = 0.99914198.$$

The reference [Mil99] provides a set of independent linear programs, a solution of which would synthesize the polyhedron $S(G, w)$, which is λ -contractive with respect to closed loop system. With AMNs, we have an alternate, fully automatic method to validate piecewise affine Lyapunov functions for stability analysis of LTI discrete time systems with saturated closed loop control inputs.

3.3 Nonconvex optimal control

This example analyzes finite horizon optimal control problems with affine (both convex and nonconvex) control constraints. Consider the piecewise affine discrete linear system

$$x(t+1) = Ax(t) + Bu(t),$$

where

$$A = \begin{bmatrix} 1 & \delta t \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\delta t^2}{2m} \\ \frac{\delta t}{m} \end{bmatrix},$$

and initial and goal states

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1(N) \\ x_2(N) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The control magnitude is bounded above and below by

$$0.2/T^{\text{tot}} \leq \|u(t)\|_1 \leq 1/T^{\text{tot}}, \quad T^{\text{tot}} = 7.5\text{s}.$$

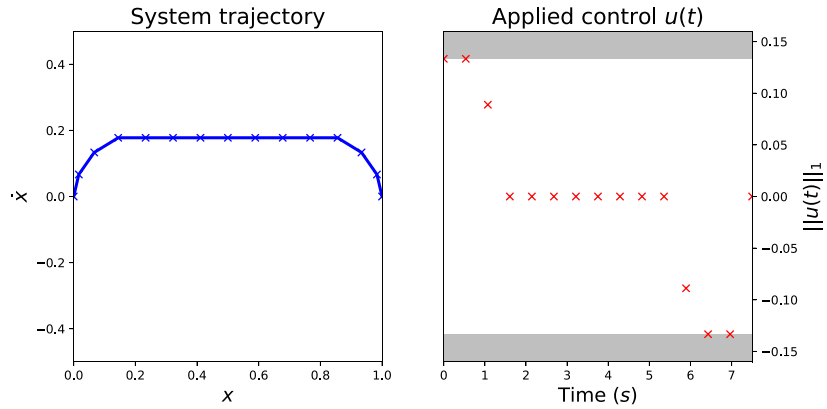
Note that the upper bound is a convex constraint, while the lower bound is not. The goal is to reach the goal state while minimizing the objective

$$J = \sum_{t=0}^{N-1} \|u(t)\|_1.$$

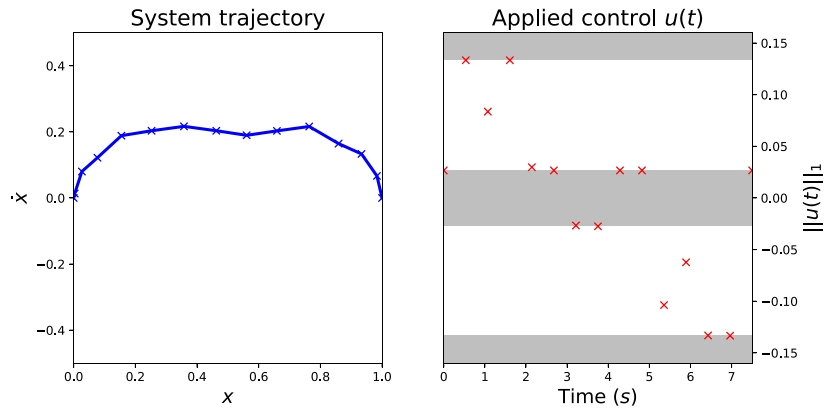
We solve the (nonconvex) optimization problem

$$\begin{aligned} & \text{minimize} && \sum_{t=0}^{N-1} \|u(t)\|_1 \\ & \text{subject to} && x(t+1) = Ax(t) + Bu(t), \quad t = 0, \dots, N-1 \\ & && 0.2/T^{\text{tot}} \leq \|u(t)\| \leq 1/T^{\text{tot}}, \quad t = 0, \dots, N \\ & && x(0) = (0, 0), \quad x(N) = (1, 0). \end{aligned}$$

with AMNET. Figure 9 summarizes the results. Note that the lower bounds are enforced for the nonconvex optimization problem, leading to a control schedule that obeys the lower bound on the control magnitude by never coasting with $u(t) = 0$; such a control schedule is not obvious from the optimal schedule for the convex problem, but it is provably optimal.



(a) Convex (CVXPY)



(b) Combined (AMNET)

Figure 9: Illustration of nonconvex optimization capabilities of AMNET. Optimal control trajectories for the convex problem (upper control bound only) using CVXPY [DB16] and combined (upper and lower control bounds) using AMNET. Shading indicates disallowed control inputs.

3.4 Characterizing classifier robustness

To illustrate the use of AMNs in other neural network applications, we used TENSORFLOW to train a simple and small neural network classifier on the popular MNIST handwritten digit dataset [AAB⁺15, LBBH98]. To make conversion and verification of the corresponding AMN manageable, the dimension of the training data was first reduced from 784 (28×28 handwritten digit images) to 40 using PCA whitening. The resulting classifier was automatically converted to an AMN, and its associated input-output relationship encoded as SMT constraints with our accompanying AMNET modeling toolbox. The baseline, 20-unit hidden layer AMN $\varphi^{\text{MNIST}} : \mathbf{R}^{40} \rightarrow \mathbf{R}^{10}$, with ReLU nonlinearities between the layers, achieved a classification rate of 0.8736 on the test data set.

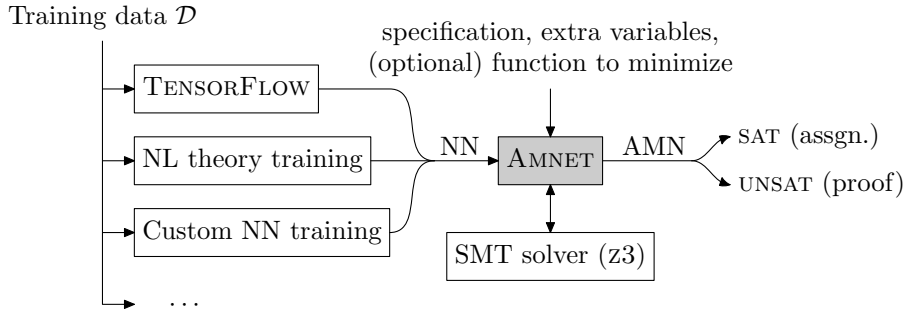


Figure 10: Using AMNET to convert a NN to an AMN.

The procedure to convert a NN with piecewise affine nonlinearities to an AMN is entirely mechanical (§1.5), and importantly, indifferent to the specific optimization algorithm (e.g., gradient descent, batching, regularization) used to train the original classifier. Therefore, once the weights and biases of the NN are encoded in SMT, formal properties of the original NN can be readily verified using our toolbox, see Figure 10.

A perturbation $\varepsilon = (\varepsilon_1, \dots, \varepsilon_5, 0, \dots, 0) \in \mathbf{R}^{40}$ was created to act on the 5 most significant components of the dimensionality reduced input, $X_{\text{amn}} = X_{\text{pca}} + \varepsilon$. Each dimension of the perturbation was constrained to $-3 \leq \varepsilon_i \leq 3$. The output layer of φ was constrained to produce a misclassification of ‘5’ by introducing the equality constraint

$$\max(\varphi^{\text{MNIST}}(X_{\text{amn}})) = \varphi^{\text{MNIST}}(X_{\text{amn}})_6.$$

on the final classification layer. The z3 SMT solver was used to find a solution to these constraints, shown below. The resulting perturbed image was recovered by $X^T = V^T X_{\text{amn}}$.

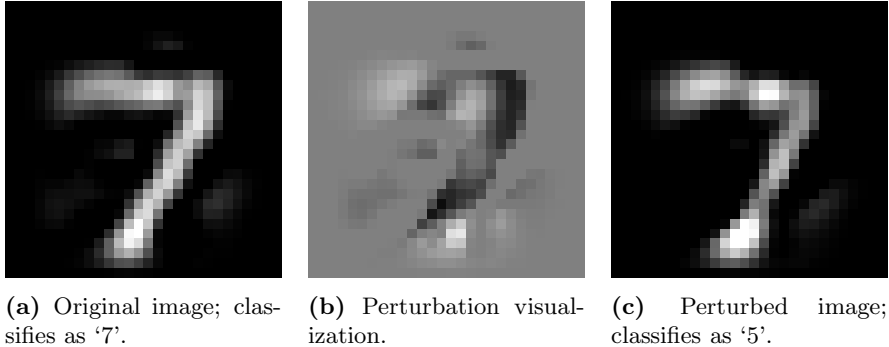


Figure 11: Perturbation on MNIST images.

4 Conclusion and extensions

In this paper, we introduced the concept of an affine multiplexing network (AMN), which is a relation built up using two fundamental building blocks: a multiplexing function (μ), and an affine transformation (α). By restricting to these two building blocks, it is possible to formally encode the relation in linear arithmetic, and therefore operate on it with existing SMT and MIP solvers. We applied this framework to nonlinear controller verification, and synthesis of stability proofs of neural networks in the loop. We also introduced the software package AMNET to make modeling with AMNs simple. Many extensions are possible, including the following subjects of future work:

- **Conic enable condition** Instead of $z \leq 0$, we can take $z \in \mathcal{K}$ to be an arbitrary cone. The multiplexer nonlinearity becomes

$$\mu_{\mathcal{K}}(x, y, z) = \begin{cases} x, & \text{if } z \in \mathcal{K}, \\ y, & \text{otherwise.} \end{cases}$$

The standard nonlinearity is the same as $\mu(x, y, z) = \mu_{-\mathbf{R}}(x, y, z)$. The verification problem translates to conic existential conditions, which can be tractably computed with a convex modification to the DPLL algorithm in SMT solvers.

- **Simulating simple programs** Many programs can be written as if-then-else networks. In general, as long as a closed loop system can be written as $x(t+1) = \varphi(x(t))$, then AMNs can be used to model them.
- **Continuous time dynamics** Dynamics governed by differential equations $\dot{x} = \varphi(x)$ can be treated just as well as discrete time dynamics governed by difference equations $x(t+1) = \varphi(x(t))$, under regularity assumptions the function φ , which guarantee existence and uniqueness of solutions.

- **Path planning and Model Predictive Control** Since we can represent constraints $x(t + 1) = \varphi(x(t))$, $t = 0, \dots, N - 1$ over a time horizon N , we can do optimal path planning for any (potentially nonlinear) system, including switched systems. This can help us answer questions about whether a particular policy for a small system (e.g., a switched power converter) is optimal or only near optimal, and to come up with novel solutions for nonlinear systems.
- **Vision in the loop** As long as every component of a closed loop system is (or can be modeled as) an AMN, formal verification of vision in the loop systems can in principle be attempted. See Figure 12.

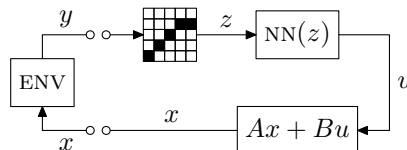


Figure 12: Vision system in the loop with AMN environment model ENV.

- **Convolutional linear units** Convolutional units are simply affine units with a special case Toeplitz structure, which admit specialized algorithms. We can take advantage of special structure forming convolution equality constraints like $y = c * x$, where c is the convolution kernel.

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